The Fourier Transform and its applications

Jean Baptiste Joseph Fourier,
* March 21st, 1768, † May 16th, 1830

goals:
- present the Fourier Theory and its applications in a concise overview
- show its fundamental relevance for many field of physics
- encourage use of FT-based reasoning and approaches

addresses:
- students of advanced semesters (post Vordiplom, master-level)

literature:
The Fourier Transform and its applications

contents:

- mathematical basis
- signal processing: filtering, sampling, reconstruction
- statistics: mean/variance, central limit theorem, noise, correlation and drift
- optics: diffraction, antenna theorem, interferometry
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contents:

• mathematical basis
  – definition, properties, cos-, sine-transform
  – convolution, autocorrelation, power spectrum
  – often used functions (hat, sinc, ...), δ-function
  – FT theorems, properties of moments
  – periodic functions, Fourier series
  – discrete Fourier transform, FFT
  – FT in higher dimensions, cylinder-, spherical coordinates

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  – filtering
  – sampling (Nyquist theorem)
  – reconstruction
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The Fourier Transform and its applications

contents:

• mathematical basis

• signal processing: filtering, sampling, reconstruction

• statistics
  – mean/variance in time and frequency domain,
  – Heisenberg relation
  – central limit theorem
  – noise, correlation and drift

• optics: diffraction, antenna theorem, interferometry
fractional-Brownian-motion (fBm) time series

drift, natural shapes: fractional Brownian motion
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- mathematical basis
- signal processing: filtering, sampling, reconstruction
- statistics: mean/variance, central limit theorem, noise, correlation and drift
- optics
  - diffraction theory
  - antenna theorem
  - interferometry
interferometry:
left: baseline-tracks of telescope array with earth rotation
right: point-spread-function

diffraction image of circular aperture
1. Mathematical Basics

Note: no rigorous proofs here, rather a “pragmatic” approach

1.1. Definition of the **Fourier Transform** (FT)

Function \( f(x) \)

Fourier Transform

\[
F(s) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i xs} \, dx \quad \text{("minus-\(i\" transform\)}
\]

note on notation: function \( f \) lower capital; Fourier transform \( F \) upper capital

- spatial: \( x \leftrightarrow s \quad f(x) \leftrightarrow F(s) \)
- time: \( t \leftrightarrow \nu \quad f(t) \leftrightarrow F(\nu) \)

Backtransform, **inverse Fourier Transform**

\[
f(x) = \int_{-\infty}^{\infty} F(s) e^{2\pi i xs} \, ds \quad \text{("plus-\(i\"-transform\) (prove given below)}
\]

**Interpretation:**

Fourier Transform: filters out the (complex) amplitude \( F(s) = |F(s)| e^{i\varphi} \) of \( e^{-2\pi i xs} \) -component, i.e. oscillation with frequency \( s \)

Back-Transform: express \( f(x) \) as sum over \( e^{2\pi i xs} \) -oscillations with (complex) amplitudes, i.e. amplitudes and phase \( F(s) = |F(s)| e^{i\varphi} \)
Remark: usual abbreviation: \( \omega = 2\pi \nu \) (time: angular frequency) or \( k = 2\pi \sigma \) (spatial: wavenumber)

gives FT: 
\[
\hat{F}(\omega) = F\left(\frac{\omega}{2\pi}\right) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \frac{\omega}{2\pi} t} dt = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt
\]

and back-Trafo 
\[
f(t) = \int_{-\infty}^{\infty} F(\nu) e^{2\pi i \nu t} d\nu = \int_{-\infty}^{\infty} F\left(\frac{\omega}{2\pi}\right) e^{i\omega t} \frac{d\omega}{2\pi} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}(\omega) e^{i\omega t} d\omega
\]
or, for symmetry, define:

\[ \tilde{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt \]

and back-transform

\[ f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{F}(\omega) e^{i\omega t} \, d\omega \]

All (1), (2), and (3) are common notations, used in different fields and contexts; we prefer the first one.

Other notations often used:

- instead of \( F(s) : \tilde{f}(s) \) or \( \tilde{f}(s) \).

  This is useful to note down certain relations, e.g. \( \tilde{f} \cdot \tilde{g} = \tilde{f * \tilde{g}} \) (FT of convolution, see below)

- \( F(s) = FT\{f(x)\} \), operator \( FT \) generates Fourier-Trafo of function \( f(x) \)
1.2. Existence of the FT

intuitively, and from physical insight: each time series has spectrum, i.e. contains a distribution of frequencies

this applies to all functional dependencies occurring in the real world physics

but:

even simple examples (which don't as such occur in the real world) show problems

- Sine, Cosine: \( a_0 \sin(2\pi \nu t), a_0 \cos(2\pi \nu t) \) (needs to be switched on and off!)
- Heavyside Step-Function: \( H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \) (needs to be switched off!)
- Delta-Funktion, Impuls-Fkt.: \( \delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases} \) (more specifically \( \int_{-\infty}^{\infty} \delta(t) \, dt = 1 \)) (never gets infinitely sharp!)

do not have FT in the proper sense

**Fourier transform in the proper sense**

the Fourier-Transform exists (i.e. the Fourier-Integral converges for all values of \( s \)) if:

1. \( |f(x)| \) is integrable, i.e. \( \int_{-\infty}^{\infty} |f(x)| \, dx \) exists
2. \( f(x) \) has only finite discontinuities
3. and \( f(x) \) shows “limited variations” (Lipshitz-Bedingung)
improper Fourier Transform

Many functions allow FT only in the “improper” sense: **FT in the limit:**

If \( \int_{-\infty}^{\infty} |f(x)| \, dx \) does not exist, one considers a modified function \( f_\alpha(x) = e^{-\alpha x^2} f(x) \).

**Note:** \( \lim_{\alpha \to 0} f_\alpha(x) = f(x) \); \( \alpha \ll 1 \): slowly switching \( f(x) \) on and off

If \( \int_{-\infty}^{\infty} \left| e^{-\alpha x^2} f(x) \right| \, dx \) exists, consider the series of varying \( \alpha \), \( \alpha \to 0 \)

\[
F_\alpha(s) = \int_{-\infty}^{\infty} e^{-2\pi i x s} e^{-\alpha x^2} f(x) \, dx
\]

and name \( \lim_{\alpha \to 0} F_\alpha(s) = F(s) \) the (improper) Fourier Transform

Note: such improper FTs are not really functions, but distributions (see below).
1.3 properties of the FT

1.3.0 linearity

with \( h(x) = a \, f(x) + b \, g(x) \), we get \( H(s) = a \, F(s) + b \, G(s) \)
(proof: linearity of integration)

1.3.1 symmetries

e.g. \( f(x) \) real valued, i.e. \( f^*(x) = f(x) \), and with the FT \( F(s) = \int e^{-2\pi i xs} f(x) \, dx \)
we get: \( F^*(s) = \int e^{2\pi i xs} f^*(x) \, dx = \int e^{-2\pi i x(-s)} f(x) \, dx = F(-s) \), i.e. \( F(s) \) is hermitean.

or, more specifically, separating \( F(s) \) into real and imaginary part:
\[
F(s) = \Re F(s) + i \Im F(s) \\
F^*(s) = \Re F(s) - i \Im F(s) \\
F(-s) = \Re F(-s) + i \Im F(-s)
\]
and hence
\[
\Re F(s) = \Re F(-s) \text{, i.e. } \Re F(s) \text{ is even} \\
\Im F(s) = -\Im F(-s) \text{, i.e. } \Im F(s) \text{ is odd; q.e.d.}
\]
Thus:
\( f(x) \) real valued \( \iff \) \( F(s) \) hermitean

and vice versa:
\( f(x) \) hermitean \( \iff \) \( F(s) \) real valued

Similarly: \( f(x) \) imaginary, i.e. \( f^*(x) = -f(x) \):
we get \( F^*(s) = \int e^{2\pi i xs} f^*(x) \, dx = -\int e^{-2\pi i x(-s)} f(x) \, dx = -F(-s) \),
Thus:
\( f(x) \) imaginary \( \iff \) \( F(s) \) anti-hermitean

and vice versa
\( f(x) \) anti-hermitean \( \iff \) \( F(s) \) imaginary
and last: \( f(x) \) even, i.e. \( f(x) = f(-x) \):

we get

\[
F(s) = \int_{-\infty}^{\infty} e^{-2\pi i x s} f(x) \, dx = -\int_{-\infty}^{\infty} e^{2\pi i u s} f(u) \, du
\]

\[
= \int_{-\infty}^{\infty} e^{-2\pi i u(-s)} \, f(u) \, du = F(-s)
\]

Thus:

\[
\begin{align*}
f(x) \text{ even} & \iff F(s) \text{ even} \\
f(x) \text{ odd} & \iff F(s) \text{ odd}
\end{align*}
\]
Note:

any function \( f(x) \) can be separated into its odd and even parts, \( o(x) \) and \( e(x) \):

with \( e(x) = e(-x) \), \( o(x) = -o(-x) \) we have:
\[
\begin{align*}
  f(x) &= e(x) + o(x) \quad \Rightarrow \quad e(x) = \frac{1}{2} \left( f(x) + f(-x) \right) \\
  f(-x) &= e(x) - o(x) \quad \Rightarrow \quad o(x) = \frac{1}{2} \left( f(x) - f(-x) \right).
\end{align*}
\]

similarly: separation into real and imaginary part:
\[
\begin{align*}
  f(x) &= \Re f(x) + i \Im f(x) \quad \Rightarrow \quad \Re f(x) = \frac{1}{2} \left( f(x) + f^*(x) \right) \\
  f^*(x) &= \Re f(x) - i \Im f(x) \quad \Rightarrow \quad \Im f(x) = \frac{1}{2i} \left( f(x) - f^*(x) \right).
\end{align*}
\]

With this, we can separate the hermitean and anti-hermitean parts:
\[
\begin{align*}
  f(x) &= e(x) + o(x) = \Re e(x) + i \Im e(x) + \Re o(x) + i \Im o(x) \\
  &= \left[ \Re e(x) + i \Im o(x) \right] + \left[ \Re o(x) + i \Im e(x) \right] = h(x) + a(x) \\
  f^*(-x) &= \Re e(x) - (-) i \Im o(x) + \left[ -\Re o(x) - i \Im e(x) \right] = h(x) - a(x)
\end{align*}
\]

so that
\[
\begin{align*}
  h(x) &= \frac{1}{2} \left( f(x) + f^*(-x) \right) \\
  a(x) &= \frac{1}{2} \left( f(x) - f^*(-x) \right)
\end{align*}
\]
to summarize:

**Symmetry properties of FT**

\[
\begin{align*}
  f(x) &= e(x) + o(x) = \Re f(x) + i \Im f(x) = h(x) + a(x) \\
  F(s) &= E(S) + O(s) = H(s) + A(s) = \Re F(s) + i \Im F(s)
\end{align*}
\]

or

\[
\begin{align*}
  f(x) &= \Re e(x) + i \Im e(x) + \Re o(x) + i \Im o(x) \\
  F(s) &= \Re E(s) + i \Im E(s) + i \Im O(s) + \Re O(s)
\end{align*}
\]

**proof:**  
e.g.  
\[
\begin{align*}
  \text{FT} \{ e(x) \} &= \overline{e(x)} = \frac{1}{2} \left[ f(x) + f(-x) \right] = \frac{1}{2} \left[ \overline{f(x)} + \overline{f(-x)} \right] \\
  &= \frac{1}{2} \left[ \int_{-\infty}^{+\infty} e^{-2\pi i x s} f(x) \, dx + \int_{-\infty}^{+\infty} e^{-2\pi i x s} f(-x) \, dx \right] \\
  &= \frac{1}{2} \left[ F(S) + \int_{-\infty}^{+\infty} e^{-2\pi i u(-s)} f(u) \, d u \right] \\
  &= \frac{1}{2} \left[ F(s) + F(-s) \right] = E(s)
\end{align*}
\]

q.e.d.
1.3.2 FT of the complex conjugate function
for the Fourier Transform of $f^*(x)$ we get:

$$\text{FT}\left(f^*(x)\right)=\overline{f^*(x)}=\int e^{-2\pi i x s} f^*(x) \, dx$$
$$=\int \left|e^{-2\pi i x(-s)}\right|^* f^*(x) \, dx=F^*(-s)$$

with this, and with $\text{FT}\left(f(-x)\right)=F(-s)$, we can derive all the above symmetry relations:

e.g. $\text{FT}\left(\Re f(x)\right)=\text{FT}\left(\frac{1}{2}[f(x)+f^*(x)]\right)=\frac{1}{2}[F(S)+F^*(-s)]=H(s)$,
or

$$\text{FT}\left(a(x)\right)=\text{FT}\left(\frac{1}{2}[f(x)-f^*(-x)]\right)=\frac{1}{2}[F(S)-F^*(s)]=i\Im F(s)$$

etc.

Thus, all symmetry relations above can be derived from the (easy to memorize) three relations:

1) $\text{FT}\left|f(x)\right|=F(s)$  
2) $\text{FT}\left|f(-x)\right|=F(-s)$  
3) $\text{FT}\left|f^*(x)\right|=F^*(-s)$
1.4 Sine- and Cosine-Transformation

Recall Euler's relation: $e^{i\alpha} = \cos \alpha + i \sin \alpha$. Inserting this into the definition of the FT, we get:

$$F(s) = \text{FT}\{f(x)\} = \int_{-\infty}^{+\infty} e^{-2\pi i xs} f(x) \, dx = \int_{-\infty}^{+\infty} \cos(2\pi xs) f(x) \, dx - i \int_{-\infty}^{+\infty} \sin(2\pi xs) f(x) \, dx$$

$$= \int_{0}^{\infty} f(x) \cos(2\pi xs) \, dx + \int_{-\infty}^{0} f(x) \cos(2\pi xs) \, dx$$

$$- i \int_{0}^{\infty} f(x) \sin(2\pi xs) \, dx - i \int_{-\infty}^{0} f(x) \sin(2\pi xs) \, dx$$

Substituting in the second and forth expression: $\int_{-\infty}^{0} x \rightarrow \int_{0}^{\infty} -u \, du$, we get:

$$= \int_{0}^{\infty} f(x) \cos(2\pi xs) \, dx + \int_{0}^{\infty} f(-u) \cos(2\pi us) \, du$$

$$- i \int_{0}^{\infty} f(x) \sin(2\pi xs) \, dx + i \int_{0}^{\infty} f(-u) \sin(2\pi us) \, du$$

$$= \int_{0}^{\infty} \left[ f(x) + f(-x) \right] \cos(2\pi xs) \, dx - i \int_{0}^{\infty} \left[ f(x) - f(-x) \right] \sin(2\pi xs) \, dx$$

$$= 2 \int_{0}^{\infty} e(x) \cos(2\pi xs) \, dx - i 2 \int_{0}^{\infty} o(x) \sin(2\pi xs) \, dx$$

We define:

**Cosine-Transformation**

$$\text{CT}\{f(x)\} = 2 \int_{0}^{\infty} f(x) \cos(2\pi xs) \, dx$$

**Sine-Transformation**

$$\text{ST}\{f(x)\} = 2 \int_{0}^{\infty} f(x) \sin(2\pi xs) \, dx$$

note: only $x > 0$ counts in CT and ST!
\[ F(s) = \text{FT} \left( f(x) \right) = \text{CT} \left( e(x) \right) + i \text{ST} \left( o(x) \right) \]

**Note:** as \( e(x) \) and \( o(x) \) need to be defined only for \( x>0 \), the continuation to \( x<0 \) being given by symmetry \( (e(-x)=e(x); o(-x)=-o(x)) \), the Fourier transform carries the full information on \( f(x) \), both for \( x>0 \) and \( x<0 \), although the integration in the CT and ST only use \( e(x) \) and \( o(x) \) for \( x>0 \).

We can thus, like always, get \( f(x) \) back from \( F(s) \) through the "+i" transform, giving:
\[
f(x) = \text{FT}_{+i} \left[ F(s) \right] = \text{CT} \left[ E(s) \right] + i \text{ST} \left[ O(s) \right]
\]

**i.e. Fourier Backtransform \( \iff \) Cosine-Transform of the even and Sine-Transform of the odd part of \( F(s) \)**

**Note:** in the CT, ST-world (no FT known),
given \( F(s) \), calculate \( E(s) \) and \( O(s) \), from this, through CT resp. ST, calculate \( f(x) \)

**Symmetries of Cosine- and Sine-Transform:** with \( \text{FT} \left( f(x) \right) = \text{CT} \left[ f(x) \right], \text{FT} \left( f(x) \right) = \text{ST} \left[ f(x) \right] \),
we have:
\[
\begin{align*}
F_C(-s) &= F_C(s) & \text{Cosine-Transform is even in } s \\
F_S(-s) &= -F_S(s) & \text{Sine-Transform is odd in } s
\end{align*}
\]
only the behaviour of $f(x)$ for $x \geq 0$ enters into to Cosine- and Sine-Transform

the symmetries of $F_C(s)$ and $F_S(s)$ imply that they are fully determined by their $s > 0$ behavior

Thus, only the $x > 0$ behavior of an arbitrary function $f(x)$ determines its $F_C(s)$ and $F_S(s)$

Consider a new function $f_H(x)$:

$$f_H(x) = \begin{cases} 
  f(x), & x \geq 0 \\
  0, & x < 0 
\end{cases} = f(x) \, H(x),$$

obtained by "cutting-off" the negative part of $f(x)$, i.e. by multiplying $f(x)$ with the Heavyside step-function defined above.

Obviously, it has the same Cosine- and Sine transform as $f(x)$ itself:

$$F_{S,H}(s) = \text{ST} \{ f_H(x) \} = 2 \int_{0}^{\infty} f_H(x) \sin(2\pi x s) \, dx$$

$$= 2 \int_{0}^{\infty} f(x) \sin(2\pi x s) \, dx = \text{ST} \{ f(x) \} = F_S(s)$$

and similarly $F_{C,H}(s) = F_C(s)$.

Now, $e_H(x) = \frac{1}{2} [ f_H(x) + f_H(-x) ] = \begin{cases} 
  \frac{1}{2} f(x), & x \geq 0 \\
  \frac{1}{2} f(-x), & x < 0 
\end{cases}$

and $o_H(x) = \frac{1}{2} [ f_H(x) - f_H(-x) ] = \begin{cases} 
  \frac{1}{2} f(x), & x \geq 0 \\
  \frac{1}{2} f(-x), & x < 0 
\end{cases}$
\[
\frac{1}{2} \left[ \text{CT}(f(x)) - i \text{ST}(f(x)) \right].
\]

**Backtransform of Cosine- and Sine-transform**

With this, we now consider an arbitrary function \( f(x) \), defined only at \( x > 0 \) and with

\[
\text{CT}(f(x)) = F_C(s) \\
\text{ST}(f(x)) = F_S(s)
\]

Note that \( F_C(s) \) and \( F_S(s) \), given their symmetry, have to be taken into account only at positive frequencies \( s > 0 \).

We claim that

the back-transform of the Cosine transform is the Cosine transform:

\[
f(x) = 2 \int_0^\infty F_C(s) \cos(2\pi xs) \, ds
\]

and the back-transform of the Sine transform is the Sine transform:

\[
f(x) = 2 \int_0^\infty F_S(s) \sin(2\pi xs) \, ds
\]
Proof:

1) for the Cosine back-trafo:

define an even function for all \( x \): \( f_1(x) = f(|x|) = \begin{cases} f(x), & x \geq 0 \\ f(-x), & x < 0 \end{cases} \), i.e. \( e_1(x) = f_1(x) \), \( o_1(x) = 0 \).

Its FT is

\[
\text{FT}(f_1(x)) = F_1(s) = \text{CT}[e_1(x)] - i \text{ST}[o_1(x)] = \text{CT}[f(x)] = F_C(s),
\]

which is also even, being a pure cosine transform:

\( F_1(-s) = F_1(s) \), resp. \( E_1(s) = F_1(s) \), \( O_1(s) = 0 \)

Its back-transform reproduces \( f_1(x) \):

\[
f_1(x) = \text{FT}_{+i} \left[ F_1(s) \right] = \text{CT}[E_1(s)] + i \text{ST}[O_1(s)] = \text{CT}[\text{CT}[f(|x|)]]
\]

and for \( x \geq 0 \), where \( f(x) \) is defined: \( f(x) = f_1(x) = \text{CT}[\text{CT}[f(x)]] = \text{CT}[F_C(s)] \) q.e.d.

2) similarly for the Sine backtrafo, but with the odd function \( f_2(x) = \begin{cases} f(x), & x \geq 0 \\ -f(-x), & x < 0 \end{cases} \):

\[
\text{FT}(f_2(x)) = F_2(s) = \text{CT}[e_2(x)] - i \text{ST}[o_2(x)] = -i \text{ST}[f(x)] = -i F_S(s),
\]

\( f_2(x) = \text{FT}_{+i} \left[ F_2(s) \right] = \text{CT}[E_2(s)] + i \text{ST}[O_2(s)] = i(-i) \text{ST}[F_S(s)] = ST[\text{ST}[f(|x|)]] \) q.e.d.
Summary:

- Fourier-Transform
  \[ F(s) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i xs} \, dx \]
  ("minus-i" transform)

- Fourier-Backtransform
  \[ f(x) = \int_{-\infty}^{\infty} F(s) e^{2\pi i xs} \, ds \]
  ("plus-i"-transform)

- Symmetry relations:
  \[ f(x) = e(x) + o(x) = \Re f(x) + i \Im f(x) = h(x) + a(x) \]
  \[ F(s) = E(S) + O(s) = H(s) + A(s) = \Re F(s) + i \Im F(s) \]

- Cosine-, and Sine-Transform
  \[ \text{CT} \left| f(x) \right| = 2 \int_{0}^{\infty} f(x) \cos(2\pi xs) \, dx \]
  \[ \text{ST} \left| f(x) \right| = 2 \int_{0}^{\infty} f(x) \sin(2\pi xs) \, dx \]

  which are their own back-transforms

- Fourier Transform
  \[ F(s) = \text{FT} \left| f(x) \right| = \text{CT} \left| e(x) \right| - i \text{ST} \left| o(x) \right| \]